Using Counter-Examples in Teaching Calculus

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ABSTRACT: The paper deals with a practical issue encountered by many lecturers teaching first-year university Calculus. A big proportion of students seem to be able to find correct solutions to test and exam questions using familiar steps and procedures. Yet they lack deep conceptual understanding of the underlying theorems and sometimes have misconceptions. In order to reduce or eliminate misconceptions, and for deeper understanding of the concepts involved, the students were given the incorrect mathematical statements and were asked to construct counter-examples to disprove the statements. More than 600 students from 10 universities in different countries were questioned regarding their attitudes towards the method of using counter-examples for eliminating misconceptions and deeper conceptual understanding. The vast majority of the students reported that the method was very effective and made learning mathematics more challenging, interesting and creative.

INTRODUCTION AND FRAMEWORK

In the information age an important ability is to analyse given information and make a quick decision on whether it is true or false. A counter-example is an example that shows that a given statement (conjecture, hypothesis, proposition, rule, theorem) is false. It only takes one counter-example to disprove a statement. Counter-examples play an important role in mathematics and other subjects. They are a powerful and effective tool for scientists, researchers and practitioners. They are good indicators showing that the suggested hypothesis or chosen direction of research is wrong. Before trying to prove the conjecture or hypothesis it is often worth to look for a possible counter-example. It can save lots of time and efforts. So we decided to introduce/remind of this powerful method to our students. Creating examples and counter-examples is neither algorithmic nor procedural and requires advanced mathematical thinking which is not often taught at school [15, 19, 20]. As Seldens write 'coming up with examples requires different cognitive skills from carrying out algorithms – one needs to look at mathematical objects in terms of their properties. To be asked for an example can be disconcerting. Students have no prelearned algorithms to show the "correct way" [15]. Many students are used to concentrate on techniques, manipulations, familiar procedures and don't put much attention to the concepts, conditions of the theorems, properties of the functions, and to reasoning and justifications. Over recent years in some countries, partly due to extensive usage of modern technology, the proof component of the traditional approach in teaching mathematics to engineering students (definition-theorem-proof-example-application) almost disappeared. Students are used to relying on technology and sometimes lack logical thinking and conceptual understanding. Sometimes mathematics courses, especially at school level, are taught in such a way that special cases are avoided and students are exposed only to 'nice' functions and 'good' examples. This approach can create many misconceptions that can be explained by the Tall's generic extension principle: 'If an individual works in a restricted context in which all the examples considered have a certain property, then, in the absence of counter-examples, the mind assumes the known properties to be implicit in other contexts.' [18]. 'The rapid increase of information over very short periods of time is a major problem in engineering education that seems worldwide. Misconceptions or unsuitable preconceptions cause many difficulties' [2]. 'The basic knowledge, performance and conceptual understanding of the students in mathematics worsen' [3].

The main objective of the study was to check our assumptions on how effective the usage of counter-examples is for deeper conceptual understanding, eliminating students' misconceptions and developing creative learning environment in teaching university Calculus.

In this study, practice was selected as the basis for the research framework and, it was decided 'to follow conventional wisdom as understood by the people who are stakeholders in the practice' [1]. The theoretical framework was based on Piaget's notion of cognitive conflict [4]. Some studies in mathematics education at school mathematics level [5], [6] found conflict to be more effective than direct instruction. 'Provoking cognitive conflict to help students understand areas of mathematics is often recommended' [6]. Swedosh and Clark [7] used conflict in their intervention method to help undergraduate students to eliminate their misconceptions. 'The method essentially involved *showing* examples for which the misconception could be seen to lead to a ridiculous conclusion, and, having established a conflict in the minds of the students, the correct concept was taught' [7]. Another study by Horiguchi and Hirashima [14] used a similar approach in creating discovery learning environment in their mechanics classes. They *showed* counter examples to their students and considered them as a chance to learn from mistakes. They claim that for counter examples to be effective they 'must be recognized to be meaningful and acceptable and must be suggestive, to lead a learner to correct understanding' [14]. Mason and Watson [8] used a method of so-called boundary examples, which suggested creating by students examples to *correct* statements, theorems, techniques, and questions that satisfied their conditions. 'When students come to apply a theorem or technique, they often fail to check that the conditions for applying it are satisfied. We conjecture that this is usually because they simply do not think of it, and this is because they are not fluent in using appropriate terms, notations, properties, or do not recognise the role of such conditions' [8]. In our study, not the lecturers but *the students* were asked to create and show counter-examples to *the incorrect* statements, i.e. the students themselves established a conflict in their minds. The students were actively involved in creative discovery learning that stimulated development of their advanced mathematical thinking.

THE STUDY

To develop creative learning environment, enhance students' critical thinking skills, help them understand concepts and theorems' conditions better, reduce or even eliminate common misconceptions and encourage active participation in class, the students were given incorrect statements and asked to create counter-examples to prove that the statements were wrong. They had enough knowledge to do that. However, for most of the students that kind of activity was absolutely new, very challenging and even created psychological discomfort and conflict for a number of reasons. In the beginning some of the students could not see the difference between "proving" that the statement is correct by an example and disproving it by an example. It agrees with the following quotation from Seldens [15]: 'Students quite often fail to see a single counter-example as disproving a conjecture. This can happen when a counter-example is perceived as "the only" one that exists, rather than being see as generic'.

In our study we did not use 'pathological' cases. All the wrong statements given to the students were within their knowledge and often were related to their common misconceptions.

Below are 6 examples of such statements that were discussed with the students:

- 'With a continuous function, i.e. a function which has values of y which smoothly and continuously change for all values of x, we have derivatives for all values of x'.
- 'If the first derivative of a function is zero then the function is neither increasing nor decreasing'.
- 'At a maximum the second derivative of a function is negative and at a minimum positive'.
- The tangent to a curve at a point is the line which touches the curve at that point but does not cross it there.
- If F(x) is defined on [a,b] and continuous on (a,b) then for any N between F(a) and F(b) there is some point c between a and b for which F(c) = N.
- If absolute value of F(x) is continuous on (a,b) then F(x) is continuous on (a,b).

The first four out of the above 6 statements are quotations from textbooks on calculus for university students published by reputable publishers. In this paper we will not touch the issue of mistakes in textbooks and their effect on students learning mathematics. This is a sensitive issue. We can refer readers to a case study done by one of the co-authors [11]. The reason we mentioned here this fact is the importance of developing and enhancing critical thinking by students for analyzing any information - not only printed in newspapers but also in mathematics textbooks. In addition, some of such textbooks might be used as a good resource for students for finding incorrect statements on the given pages and creating counter-examples to them.

After several weeks of using counter-examples in teaching Calculus to first-year engineering students, 612 students from 10 universities in different countries were given the following questionnaire to investigate their attitudes towards the usage of counter-examples in learning/teaching. A cross-cultural approach was chosen to reduce the effect of differences in education systems, curricula, cultures and also to analyse the data from different perspectives and backgrounds.

THE QUESTIONNAIRE

Question 1. Do you feel confident using counter-examples?
a) Yes Please give the reasons:
b) No Please give the reasons:

Question 2. Do you find this method effective?
a) Yes Please give the reasons:
b) No Please give the reasons:

Question 3. Would you like this kind of activity to be a part of assessment?
a) Yes Please give the reasons:
b) No Please give the reasons:

FINDINGS FROM THE QUESTIONNAIRE

The statistics from the questionnaire are presented in the following table:

Number of Students	Question 1 Confident?		Question 2 Effective?		Question 3 Assessment?	
	Yes	No	Yes	No	Yes	No
612	116	496	563	49	196	416
100%	19%	81%	92%	8%	32%	68%

Table 1. Summary of findings from the questionnaire

The majority of the students (81%) were not familiar with the usage of counter-examples as a method of disproof. The common comments from the students who answered ‘No’ to question 1 on whether they are confident with using of counter-examples or not were as follows:

- I have never done this before;
- I am not familiar with this at all;
- I am not used to this method;
- this method is unknown to me;
- we did not learn it at school;
- I heard about it but not from my school teacher;
- we hardly created ourselves any examples at school.

The vast majority of the students (92%) found the method of using counter-examples to be effective. The common comments from the students who answered ‘Yes’ to question 2 on whether the usage of counter-examples is effective or not were as follows:

- helps me to think question deeply;
- gives more sound knowledge of the subject;
- we can understand more;
- it makes me think more effectively;
- can prevent mistakes;
- you gain a better understanding;
- it makes the problem more clear;
- it boosts self-confidence;
- it helps you retain information that you have learned;
- it is a good teaching tool;
- it teaches you to question everything;
- it makes you think carefully about the concepts and how they are applied;
- it makes you think critically;
- it supports self-control;
- it requires logical thinking, not only calculations;
- makes problems more understandable;
- it is hard but it is fun;
- it is a good way to select top students;
- I can look at maths from another angle;
- it is good not only in mathematics;
- it really forces you to think hard;
- it is not a routine exercise, it is creative.

The majority of the students (68%) did not want the questions on creating counter-examples to incorrect statements to be part of assessment in contrast to the trends pointing to the effectiveness of the method (92%). The common comments from the students who answered ‘No’ to question 3 on whether the questions on creating counter-examples be part of assessment or not were as follows:

- it is hard;
- never done this stuff before;
- confusing;
- not trained enough;
- complicated;
- not structured;
- not enough time to master it;
- you don’t know how to start;
- can affect marks.

The last comment was the most common. The majority of these students were more concerned about their test results rather than acquiring useful skills. Apparently their attitudes towards learning were not mature yet.

The students who answered ‘Yes’ (32%) provided excellent comments similar to those made on effectiveness of the method. The common comments from the students who answered ‘Yes’ to question 3 on whether the questions on creating counter-examples be part of assessment or not were as follows:

- it provokes generalised thinking about the *nature* of the processes involved, as compared to the detail of the processes;
- better performance test;
- it shows full understanding of topic;
- a good way to test students' insight;
- it is an extremely valuable skill;
- it is good to have it in assessment otherwise we will not put much attention to it;
- one can use
- one can use this method outside university;

CONCLUSIONS AND RECOMMENDATIONS

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The overwhelming statistics of the study and numerous students' comments showed that the students were very positive about the usage of counter-examples in first-year undergraduate mathematics. Many of them reported that the method of using counter-examples helped them to understand concepts better, prevent mistakes in future, develop logical and critical thinking, and made their participation in lectures more active. All these give us confidence to recommend this pedagogical strategy to our colleagues to try with their students. There could be different ways of using this strategy: giving the students a mixture of correct and incorrect statements; making a deliberate mistake in the lecture; asking the students to spot an error on a certain page of their textbook or manual; giving the students extra (bonus) marks towards their final grade for providing excellent counter-examples to hard questions during the lecture and so on. At more advanced level of mathematics Dahlberg and Housman suggest 'it might be beneficial to introduce students to new concepts by having them generate their own examples or having them decide whether teacher-provided candidates are examples or non-examples, before providing students examples and explanations' [13].

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Many students commented that creating counter-examples were closely connected with enhancing their critical thinking skills. These skills are general ones and can be used by the students in other areas of their life that have nothing to do with mathematics. The ability to create counter-examples is an important instrument of critical selection in the broader sense. Henry Perkinson, the author of the famous book 'Learning from our mistakes' [16], writes about importance of those skills for his theory of education in his recent publication [17]: 'Our knowledge is imperfect, it can always get better, improve, grow. Criticism facilitates this growth. Criticism can uncover some of the inadequacies in our knowledge, and when we eliminate them, our knowledge evolves and gets better…Education is a continual process of trial-and-error elimination. Students are fallible creators who make trial conjectures and formulate trial skills and then eliminate the errors uncovered by criticism and critical selection'.

FURTHER STUDY

We would like to extend the study to measure the effectiveness of this pedagogical strategy on the students' exam performance on the questions that require good understanding of concepts, not just manipulations and techniques. We plan to compare the performance of 2 groups of students with similar backgrounds. In one group we will extensively use counter-examples, with the other group being the control group. Then we will use statistical methods to establish whether the difference is significant or not.

We also plan to develop a database of incorrect statements related to common students' misconceptions for practising in creating counter-examples. This supplementary teaching resource could be used for both teaching and assessment.

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